

DC Josephson Effect with Fermi gases in the Bose-Einstein regime

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We show that the DC Josephson effect with ultracold fermionic gases in the BEC regime of composite molecules can be described by a nonlinear Schrödinger equation (NLSE). By comparing our results with Bogoliubov-de Gennes calculations [Phys. Rev. Lett. **99**, 040401 (2007)] we find that our superfluid NLSE, which generalizes the Gross-Pitaevskii equation taking into account the correct equation of state, is reliable in the BEC regime of the BCS-BEC crossover up to the limit of very large (positive) scattering length. We also predict that the Josephson current displays relevant beyond mean-field effects.

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I. INTRODUCTION

In the last few years several experimental groups have observed, close to a Fano-Feshbach resonance [1], the crossover from the Bardeen-Cooper-Schrieffer (BCS) state of Cooper pairs to the Bose-Einstein condensate (BEC) of molecular dimers in ultra-cold two-hyperfine-components Fermi vapors of ⁴⁰K atoms [2, 3, 4] and ⁶Li atoms [5, 6, 7, 8]. Few years ago the AC Josephson effect [9, 10] in atomic BECs was predicted [11] and observed [12]. AC Josephson oscillations in superfluid atomic Fermi gases have been considered theoretically by several authors [13, 14, 15, 16, 17]. Recently, Spuntarelli, Pieri and Strinati [18] have studied the DC Josephson effect [9, 10] across the BCS-BEC crossover in neutral fermions by using the extended BCS equations: they have computed the current-phase relation throughout the BCS-BEC crossover at zero temperature for a two-spin component Fermi gas in the presence of a barrier by solving the coupled Bogoliubov-de Gennes equations (BdG) [18].

In this paper we show that a simple nonlinear Schrödinger equation (NLSE) [16, 19, 20, 21] is able to reproduce the Josephson results of Spuntarelli, Pieri and Strinati [18], in the BEC side of the BCS-BEC crossover, i.e. from the deep BEC regime up to very large (positive) values of the scattering length. This NLSE is equivalent to the equations of superfluid hydrodynamics [22] with the inclusion of a gradient term [16, 19, 20, 21]. We demonstrate, in particular, that the gradient term is essential to obtain the correct current-phase Josephson relation.

As discussed in ref [18], the DC Josephson currents found by the Gross-Pitaevskii (GP) and the BdG formalisms are the same in the deep BEC regime, while for relatively large values of the scattering length the BdG equations lead to important deviations from the GP predictions. Here we will show that our NLSE formalism, equivalent to GP in the deep BEC regime, gives results which are in remarkably good agreement with the BdG method also for large, positive, values of the scattering length.

We also find that the breakdown of superfluidity, which corresponds to the maximum Josephson current across the barrier, strongly depends on the bulk equation of state embodied in the superfluid NLSE. In particular, on the basis of the Monte Carlo equation of state [20] that includes beyond mean-field effects, we predict that the critical currents are smaller than those calculated so far [18] using mean-field theories.

II. NLSE FOR SUPERFLUID FERMIONS

Inspired by the density functional theory of Helium 4 [23] and by the low-energy effective field theory of the Fermi gas in the BCS-BEC crossover [24, 25], we have recently introduced [16, 20, 21] a complex order parameter

$$\Psi(\mathbf{r}, t) = \sqrt{\frac{n(\mathbf{r}, t)}{2}} e^{i\theta(\mathbf{r}, t)}, \quad (1)$$

to describe boson-like Cooper pairs of a two-component fermionic superfluid made of atoms of mass m in the BCS-BEC crossover [22], where $n(\mathbf{r}, t)$ is the local fermion density and $\theta(\mathbf{r}, t)$ the local phase. Here $n(\mathbf{r}, t) = n_{\uparrow}(\mathbf{r}, t) + n_{\downarrow}(\mathbf{r}, t)$, with $n_{\uparrow}(\mathbf{r}, t) = n_{\downarrow}(\mathbf{r}, t)$. The normalization of $\Psi(\mathbf{r}, t)$ is such that

$$\int |\Psi(\mathbf{r}, t)|^2 d^3\mathbf{r} = \frac{N}{2}, \quad (2)$$

N being the total number of fermionic atoms. Notice that although $\Psi(\mathbf{r}, t)$ should not be interpreted as the condensate wave function [26], $\theta(\mathbf{r}, t)$ represents the condensate phase [27, 28]. The local superfluid velocity $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_\uparrow(\mathbf{r}, t) = \mathbf{v}_\downarrow(\mathbf{r}, t)$ is thus related to the phase $\theta(\mathbf{r}, t)$ by the equation [22]

$$\mathbf{v} = \frac{\hbar}{2m} \nabla \theta. \quad (3)$$

The nonlinear Schrödinger equation (NLSE) that satisfies Eq. (3) and reproduces the equations of superfluid hydrodynamics [22] in the classical limit ($\hbar \rightarrow 0$) is given by

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{4m} \nabla^2 + 2U(\mathbf{r}) + 2\mu(n(\mathbf{r}, t), a_F) \right] \Psi(\mathbf{r}, t), \quad (4)$$

where $U(\mathbf{r})$ is the external potential and $\mu(n, a_F)$ is the bulk chemical potential, i.e. the zero-temperature equation of state (EOS) of the uniform system, which depends on the fermion-fermion scattering length a_F . In fact, by using Eqs. (1) and (3), the NLSE can be written as

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\mathbf{v}) = 0 \quad (5)$$

$$m \frac{\partial}{\partial t} \mathbf{v} + \nabla \left[\frac{1}{2} m v^2 + U(\mathbf{r}) + \mu(n, a_F) + T_{QP} \right] = 0 \quad (6)$$

where

$$T_{QP} = -\frac{\hbar^2}{8m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \quad (7)$$

is a quantum pressure term containing explicitly Planck's constant \hbar . This term can be viewed as a gradient correction in the density functional theory [23] or the next-to-leading correction in a low-energy effective field theory [25]. It is important to stress that in the deep BEC regime ($a_F \rightarrow 0^+$) from Eq. (4) one recovers the familiar Gross-Pitaevskii equation for the Bose-condensed molecules made of paired fermions, where

$$\mu(n, a_F) = \frac{4\pi\hbar^2 a_{dd}(a_F)}{2m} n \quad (8)$$

with $a_{dd}(a_F)$ the dimer-dimer scattering length, which depends on the fermion-fermion scattering length a_F .

Over the full BCS-BEC crossover the bulk chemical potential can be written as [22]

$$\mu(n, a_F) = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} \left(f(y) - \frac{y}{5} f'(y) \right) \quad (9)$$

where $f(y)$ is a dimensionless universal function of the inverse interaction parameter

$$y = \frac{1}{k_F a_F} \quad (10)$$

with $k_F = (3\pi^2 n)^{1/3}$ the Fermi wavenumber and $\epsilon_F = \hbar^2 k_F^2 / (2m)$ the Fermi energy. One can parametrize $f(y)$ as follows:

$$f(y) = \alpha_1 - \alpha_2 \arctan \left(\alpha_3 y \frac{\beta_1 + |y|}{\beta_2 + |y|} \right), \quad (11)$$

where the values of the parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$, reported in Ref. [20], are fitting parameters based on asymptotics and fixed-node Monte-Carlo data [29]. We call Monte-Carlo equation of state (MC EOS) the equation $\mu = \mu(n, a_F)$ obtained from (9) and (11). Notice that Eq. (4) with Eq. (11) has been recently used to describe density profiles, collective oscillations and free expansion in the full BCS-BEC regime, finding a very good agreement with the experimental data [15, 16, 19, 20].

Within the mean-field extended BCS theory [28, 30], the bulk chemical potential μ and the gap energy Δ of the uniform Fermi gas are found by solving the following extended Bogoliubov-de Gennes equations [26, 30]

$$-\frac{1}{a_F} = \frac{2(2m)^{1/2} \Delta^{1/2}}{\pi \hbar^3} \int_0^\infty dy y^2 \left(\frac{1}{y^2} - \frac{1}{\sqrt{(y^2 - \frac{\mu}{\Delta})^2 + 1}} \right) \quad (12)$$

$$n = \frac{N}{V} = \frac{(2m)^{3/2} \Delta^{3/2}}{2\pi^2 \hbar^3} \int_0^\infty dy y^2 \left(1 - \frac{(y^2 - \frac{\mu}{\Delta})}{\sqrt{(y^2 - \frac{\mu}{\Delta})^2 + 1}} \right) \quad (13)$$

From these two coupled equations one obtains the chemical potential μ as a function of n and a_F in the full BCS-BEC crossover (see for instance Ref. [26]). Note that, contrary to the MC EOS, this mean-field theory does not predict the correct BEC limit: the molecules have scattering length $a_{dd} = 2a_F$ instead of the value $a_{dd} = 0.6a_F$ predicted by four-body and MC calculations [22]. We call mean-field equation of state (MF EOS) the equation $\mu = \mu(n, a_F)$ obtained from Eqs. (12) and (13). Clearly this MF EOS is less reliable than the MC EOS, as shown in a recent study [7].

As previously stressed, the NLSE (4) describes quite accurately static properties and low-energy collective modes of oscillation in the full BCS-BEC crossover [15, 16, 19, 20], but does not take into account the effect of pair breaking. In fact, Eq. (4) is reliable if the collective-mode wavelength λ is such that $\lambda \gg \xi$, where ξ is the healing length of the superfluid. Recently, Combescot, Kagan and Stringari [31] have suggested that

$$\xi = \frac{\hbar}{mv_{cr}} \quad (14)$$

where v_{cr} is the critical velocity of the Landau criterion for dissipation [27, 31]. According to Combescot, Kagan and Stringari [31], in the BEC regime of bosonic dimers, and in particular for $y > y_c = 0.08$, the critical velocity v_{cr} coincides with the sound velocity, i.e.

$$v_{cr} = c_s = \sqrt{\frac{n}{m} \frac{\partial \mu}{\partial n}}. \quad (15)$$

Instead, for $y < y_c = 0.08$, i.e. also at unitarity ($y = 0$) and in the BCS regime ($y < 0$), the critical velocity v_{cr} is related to the breaking of Cooper pairs through the formula

$$v_{cr} = \sqrt{\frac{\sqrt{\mu^2 + |\Delta|^2} - \mu}{m}}, \quad (16)$$

where $|\Delta|$ is the energy gap of Cooper pairs [31].

In the following section we shall show that the NLSE, Eq. (4), can be used to describe quantitatively the Josephson effect, but only on the BEC side of the BCS-BEC crossover, i.e. for $y > y_c = 0.08$. For $y < y_c = 0.08$ instead, pair breaking plays an essential role in the breakdown of the superfluid Josephson current [8, 31].

III. DIRECT CURRENT JOSEPHSON EFFECT

We apply the NLSE (4) to study the direct-current (DC) Josephson effect [9, 10]. We consider a square-well barrier

$$U(\mathbf{r}) = \begin{cases} V_0 & \text{for } |z| < \frac{L}{2} \\ 0 & \text{elsewhere} \end{cases} \quad (17)$$

which separates the superfluid into two regions, and assume a stationary solution

$$\Psi(\mathbf{r}, t) = \Phi(\mathbf{r}) e^{i\theta(\mathbf{r})} e^{-i2\bar{\mu}t/\hbar} \quad (18)$$

with constant and uniform number supercurrent

$$J = n(\mathbf{r})\mathbf{v}(\mathbf{r}) = 2\Phi(\mathbf{r})^2 \frac{\hbar}{2m} \nabla\theta(\mathbf{r}). \quad (19)$$

From the previous equation it follows $(\nabla\theta)^2 = m^2 J^2 / (\hbar^2 \Phi^4)$ and also

$$\left[-\frac{\hbar^2}{4m} \nabla^2 + \frac{m}{4} \frac{J^2}{\Phi(\mathbf{r})^4} + 2U(\mathbf{r}) + 2\mu(n(\mathbf{r}), a_F) \right] \Phi(\mathbf{r}) = 2\bar{\mu} \Phi(\mathbf{r}) \quad (20)$$

With the purpose of comparing our results with those obtained by Spuntarelli, Pieri and Strinati [18], we use the MF EOS of $\mu(n(\mathbf{r}), a_F)$ and solve the above equation by imposing a constant and uniform density \bar{n} far from the barrier region:

$$\Phi(\mathbf{r}) \rightarrow \sqrt{\frac{\bar{n}}{2}} \quad \text{for } |\mathbf{r}| \rightarrow \infty \quad (21)$$

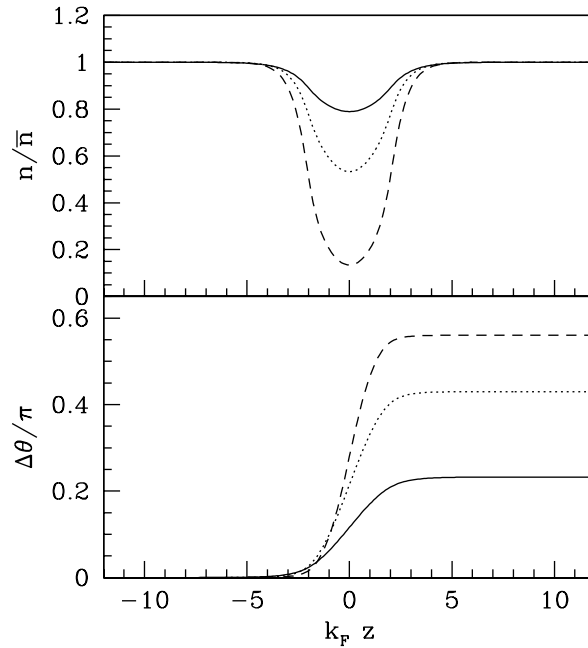


FIG. 1: Upper panel: scaled density profile n/\bar{n} in the z direction (orthogonal to the barrier). The barrier width is $L = 4/k_F$. We consider three values of the energy barrier: $V_0/\epsilon_F = 0.025$ (solid line); $V_0/\epsilon_F = 0.1$ (dotted line); $v_0/\epsilon_F = 0.4$ (dashed line). Lower panel: phase difference $\Delta\theta$ across the barrier for the same values of V_0/ϵ_F .

We integrate Eq. (20) on a 1D mesh in real space, in an interval $[0, z_{max}]$, using an imaginary time method, as described in [32], and determine $\Phi(\mathbf{r})$ by fixing the parameters of the barrier (which is located at $z = z_{max}/2$), the uniform density \bar{n} and the scattering length a_F . To compute the current-phase relation we proceed as follows. The phase difference across the barrier can be obtained from Eq. (19)

$$\Delta\theta = \frac{2mJ}{\hbar} \left[\int_0^{z_{max}} \frac{1}{2\Phi(z)^2} dz - \frac{z_{max}}{\bar{n}} \right] \quad (22)$$

We choose a value for $\Delta\theta$ and compute J , from the above equation, at each iteration in imaginary time using the actual density profile $\Phi(z)^2$. The updated value of J is then inserted into Eq. (20) for the next iteration in imaginary time. We stop the calculations when convergence is achieved, i.e. the density profile does not change anymore between two consecutive iterations.

In the upper panel of Fig. 1 we plot the scaled density profile $n(z)/\bar{n}$ of the fluid calculated for three different values of the energy barrier height. The figure shows that by increasing the energy barrier height V_0 , the dip in the profile $n(z)$ is enhanced. In the lower panel of the same figure we display the corresponding local phase difference $\Delta\theta(z)$ obtained from Eq. (22). These results are obtained by using the NLSE with the MF EOS.

The calculated relationship between the current J and the phase difference $\Delta\theta$ is shown in Fig. 2 for two values of the interaction parameter y . Here our NLSE results (solid lines) are compared with the ones of Spuntarelli, Pieri and Strinati [18], obtained by solving the full set of Bogoliubov-de Gennes equations for the quasi-particle amplitudes in the presence of the barrier (17). For both values of y shown, the agreement with the BdG theory is remarkably good. For comparison, we also show in Fig. 2 the results obtained by solving the GP equation, to which our NLSE is expected to reduce in the deep BEC regime ($y \gg 1$). Interestingly enough, we find that $y = 3$ is already in the regime well described by the GP formalism. At $y = 1$, however, large deviations of the GP curve with respect to the BdG (and NLSE as well) ones are found.

In Fig. 3 we plot the maximum J_0^{max} of the current J as a function of the inverse interaction parameter $y = 1/(k_F a_F)$, and compare our data (curves) with the results of Spuntarelli, Pieri and Strinati (symbols) [18]. Figure 3 shows that the NLSE reproduces the DC Josephson results of Ref. [18] in the BEC regime remarkably well, from the deep BEC regime ($y \gg 1$) up to very large positive values of the scattering length ($y \ll 1$). In the BCS regime

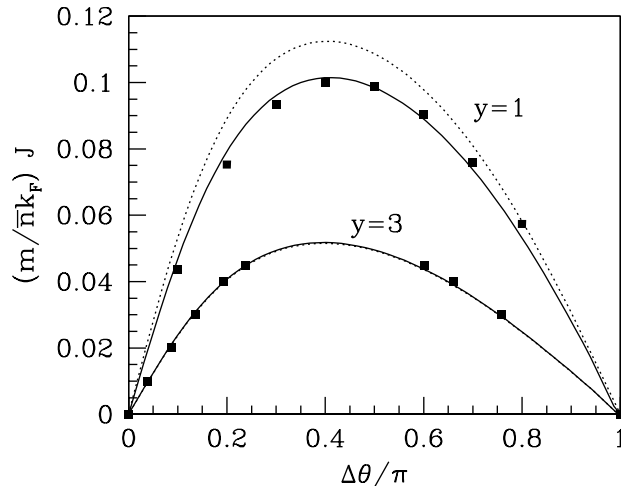


FIG. 2: DC Josephson current J vs phase difference $\Delta\theta$ for two values of the inverse interaction parameter $y = 1/(k_F a_F)$. Barrier parameters: $L = 4/k_F$, $V_0/\epsilon_F = 0.1$ (curves corresponding to $y = 1$), $L = 5.3/k_F$, $V_0/\epsilon_F = 0.05$ (curves corresponding to $y = 3$). Squares: Bogoliubov-de Gennes calculations of Ref. [18, 33]. Solid lines: NLSE with MF EOS. Dotted lines: GP results.

($y < 0$) the NLSE predictions are instead expected to be completely unreliable. This is hardly surprising since the superfluid NLSE completely neglects the effect of pair breaking.

Fig. 2 shows that deviations from the GP results of the calculated current-phase relation are found for the case $y = 1$. To further investigate to which degree the NLSE reduces to the GP case, in Fig.4 we compare the predictions of the NLSE and those of the GP equation for the maximum of the Josephson current as a function of y . It appears that for relatively large values of the interaction parameter the NLSE results dramatically deviate from the GP results (while being in good agreement with the BdG calculations). In fact, on the basis of the Eq. (15) and the pair-breaking argument of Combescot, Kagan and Stringari [31], our NLSE should be accurate from $y \ll 1$ (deep BEC limit) up to $y = y_c = 0.08$, thus also for values of y very close to the unitarity limit ($y = 0$).

Thus, the NLSE might represent a viable alternative, in the whole range of positive scattering lengths (and especially where the GP equation is not reliable, as shown in Fig.4), to the much more computationally expensive BdG approach. This is especially true in the case of 3D geometries, where the BdG method could be prohibitively costly.

As expected in our calculations we recover the Josephson equation

$$J = J_0 \sin(\Delta\theta) \quad (23)$$

in the regime of high barrier (weak-link).

We observe that, contrary to our NLSE, the classical hydrodynamics equations of Fermi superfluids [22], i.e. Eqs. (5) and (6) with $T_{QP} = 0$, cannot be used to study the Josephson effect. This is shown in Fig. 5 where we plot the current-phase diagram and compare the NLSE results (dashed lines) with the ones obtained using the classical hydrodynamic equations (dotted line). It clearly appears that the Josephson relation (23) is violated (the dotted curve does not go to zero at π) if we omit the gradient term. Moreover, the predicted current values are very different from the NLSE results.

The results discussed up to now are based on the use of the MF EOS, since we were interested in assessing the reliability of the NLSE by comparing its results with those obtained by solving the Bogoliubov-de Gennes equations [18], equivalent to a MF treatment. Of course, to give useful predictions to be compared with experiments one must use the MC EOS. In fact, relevant beyond-mean-field effects in the BEC side of the BCS-BEC crossover have been

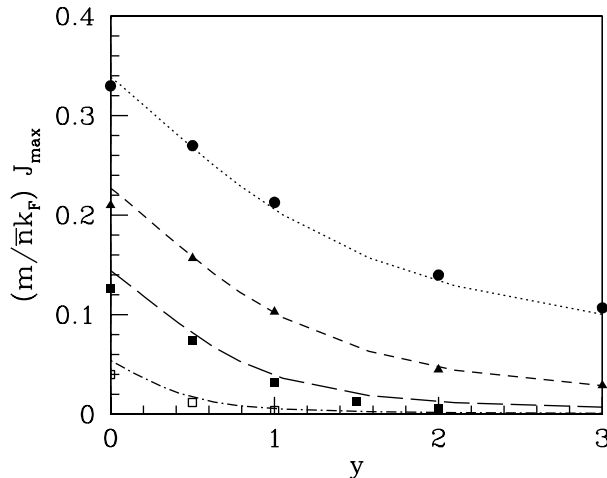


FIG. 3: Maximum Josephson current J_0^{max} vs inverse interaction parameter $y = 1/(k_F a_F)$ in the BEC region ($y > 0$). Solid curve: J_0^{max} based on pair breaking in the BCS regime [18, 31]. Other curves: superfluid NLSE. Symbols: Bogoliubov-de Gennes calculations of Ref. [18]. Four values for the energy barrier height V_0/ϵ_F are considered: 0.025, 0.10, 0.2, 0.4 (from top to bottom in the figure). The width of the barrier is $L = 4/k_F$.

predicted and observed for the density profiles and also for collective oscillations [7, 20, 22]. In a recent experiment of Miller *et al.* [8] critical velocities have been observed in an ultracold superfluid Fermi gas throughout the BCS-BEC crossover. These critical velocities, determined from the abrupt onset of dissipation when the velocity of a moving one-dimensional lattice is varied [8], are the analog of the maximum Josephson current J_0^{max} . In Fig. 6 we plot J_0^{max} as a function of the inverse interaction parameter $y = 1/(k_F a_F)$, and compare the results obtained from the superfluid NLSE by using the MC EOS (solid lines) or the MF EOS (dotted lines). The figure shows that, for a given barrier, the maximum current predicted by the MC EOS is appreciably smaller than the MF one for all values of y . Moreover, as also observed in the experiment of Miller *et al.* [8], there is a pronounced peak of J_0^{max} at unitarity ($y = 0$), which is absent in the MF curves. It is important to observe that, at the crossover, beyond-mean field effects exist not only in the bulk equation of state. In fact, we have recently shown that at unitarity ($y = 0$) the superfluid NLSE of Eq. (4) must be modified with the inclusion of an additional nonlinear term [34]. Nevertheless, this beyond-mean-field term goes to zero for a large number of atoms.

IV. CONCLUSIONS

We have introduced a NLSE equivalent to the hydrodynamic equations of Fermi superfluids plus a gradient correction. Both hydrodynamics equations and superfluid NLSE are known to reliably reproduce static properties and low-energy collective dynamics. The advantage of using the NLSE is that one can take into account also surface and shape effects, and these can be relevant for a small number of particles. In addition, the gradient term is essential to obtain the correct Josephson equation, as demonstrated in the present work. We have shown that in the study of the DC Josephson effect our NLSE works quite well at the right side (BEC regime) of the BCS-BEC crossover also for very large values of the scattering length, where the familiar Gross-Pitaevskii equation is instead unreliable. In particular, our results suggest that the superfluid NLSE is accurate from $y \ll 1$ (deep BEC limit) up to $y = y_c = 0.08$, i.e. very close to the unitarity limit ($y = 0$).

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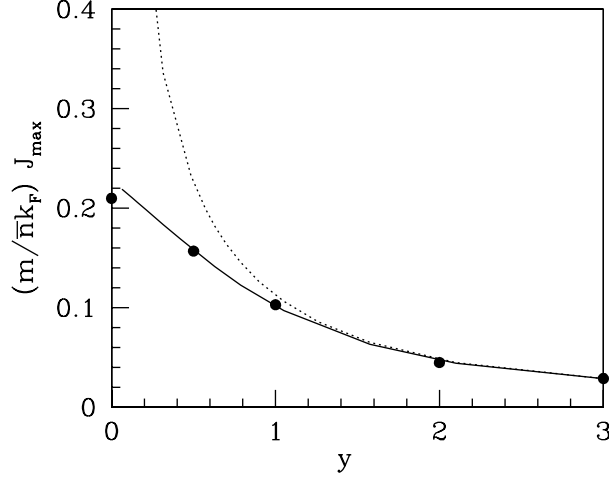


FIG. 4: Maximum Josephson current J_0^{max} vs inverse interaction parameter $y = 1/(k_F a_F)$ obtained with the NLSE. Comparison between BdG (dots) and the GP approximation (dotted lines) is made. $V_0/\epsilon_F = 0.1$ and $L = 4/k_F$.

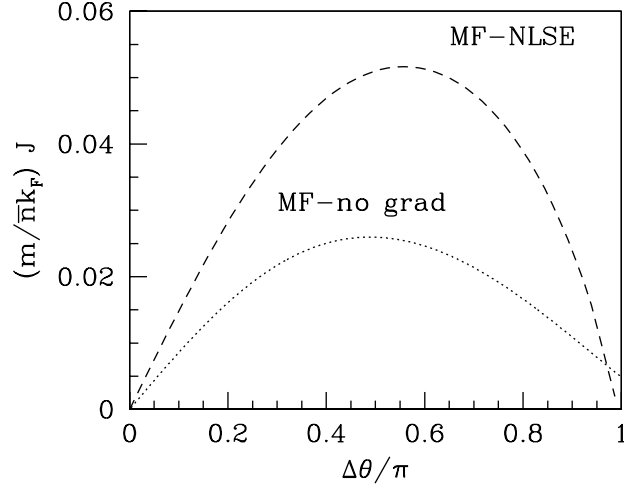


FIG. 5: Current J vs phase difference $\Delta\theta$: without the gradient term (dotted line) the Josephson-equation is violated and the result are very far from the NLSE ones (dashed line).

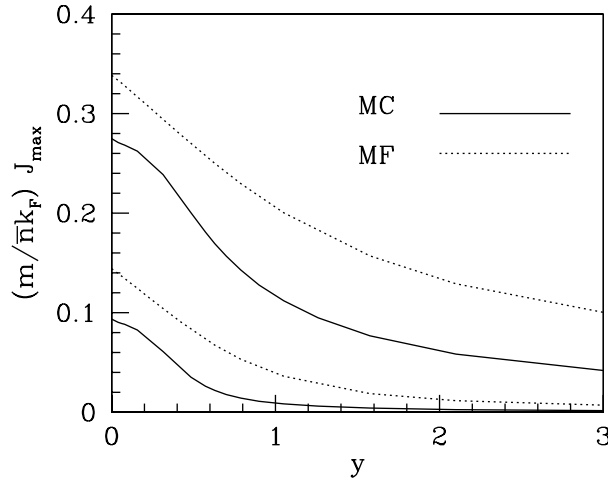


FIG. 6: Maximum Josephson current J_0^{max} vs inverse interaction parameter $y = 1/(k_F a_F)$ obtained with the NLSE. Comparison between MC EOS (solid lines) and MF EOS (dotted lines). Two values of the energy barrier height are considered: $V_0/\epsilon_F = 0.025$ (upper curves) and $V_0/\epsilon_F = 0.2$ (lower curves).

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